Calc H Notes – Section 1.4 Continuity and One-Sided Limits

Intuitively speaking, a function is continuous at a point if you can draw the graph of the function in the vicinity of that point without lifting your pencil from the paper.

Conversely, a function is discontinuous at any *x*-value where the pencil must be lifted from the paper in order to draw the graph on both sides of the point.

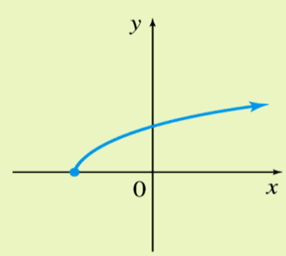
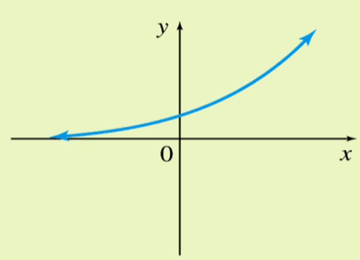
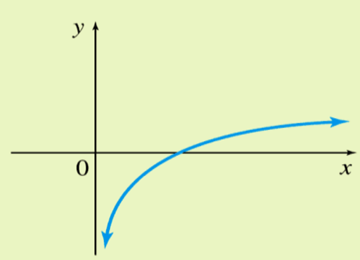
CONTINUITY AT $x = c$

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

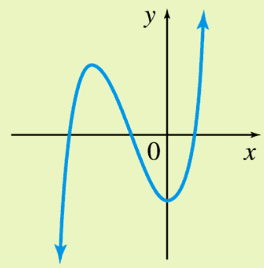
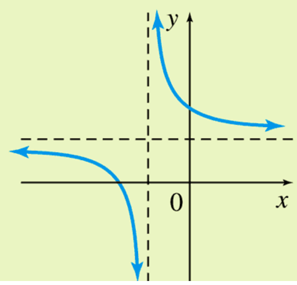
1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous** there.

Continuous Functions

Type of Function	Where It Is Continuous	Graphic Example
Root Function $y = \sqrt{ax + b}$, where a and b are real numbers, with $a \neq 0$ and $ax + b \geq 0$	For all x where $ax + b \geq 0$	
Exponential Function $y = a^x$ where $a > 0$	For all x	
Logarithmic Function $y = \log_a x$ where $a > 0$, $a \neq 1$	For all $x > 0$	

Continuous Functions

Type of Function	Where It Is Continuous	Graphic Example
Polynomial Function $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, not all 0	For all x	
Rational Function $y = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$	For all x where $q(x) \neq 0$	

Discontinuities can be removable (hole) or non-removable (vertical asymptotes and gaps). To remove the hole create a piecewise function which defines f(c)=y-coord of the hole.

One application of continuity is the Intermediate Value Theorem, which says that if a function is continuous on a closed interval $[a, b]$, the function takes on every value between $f(a)$ and $f(b)$.

- **For example, if $f(1) = -3$ and $f(2) = 5$, then f must take on every value between -3 and 5 as x varies over the interval $[1, 2]$.**

This proves that f(x) has a zero or x-intercept in the interval [1,2] since the sign of the value of f(x) changes from negative to positive at the endpts of the interval.

One-sided limits:

- **The phrase "*x* approaches *a* from the left" is written $x \to a^-$.**
  - $\lim_{x \to a^-} f(x) = 3$
- **The phrase "*x* approaches *a* from the right" is written $x \to a^+$.**
  - $\lim_{x \to a^+} f(x) = 3$

Two-sided limit:

- **Exists only if both one-sided limits exist and are the same.**

$\lim_{x \to a} f(x) = 3$

LIMIT OF A FUNCTION

Let *f* be a function and let *a* and *L* be real numbers. If

1. as *x* takes values closer and closer (but not equal) to *a* on both sides of *a*, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to *L*; and
2. the value of $f(x)$ can be made as close to *L* as desired by taking values of *x* close enough to *a*;

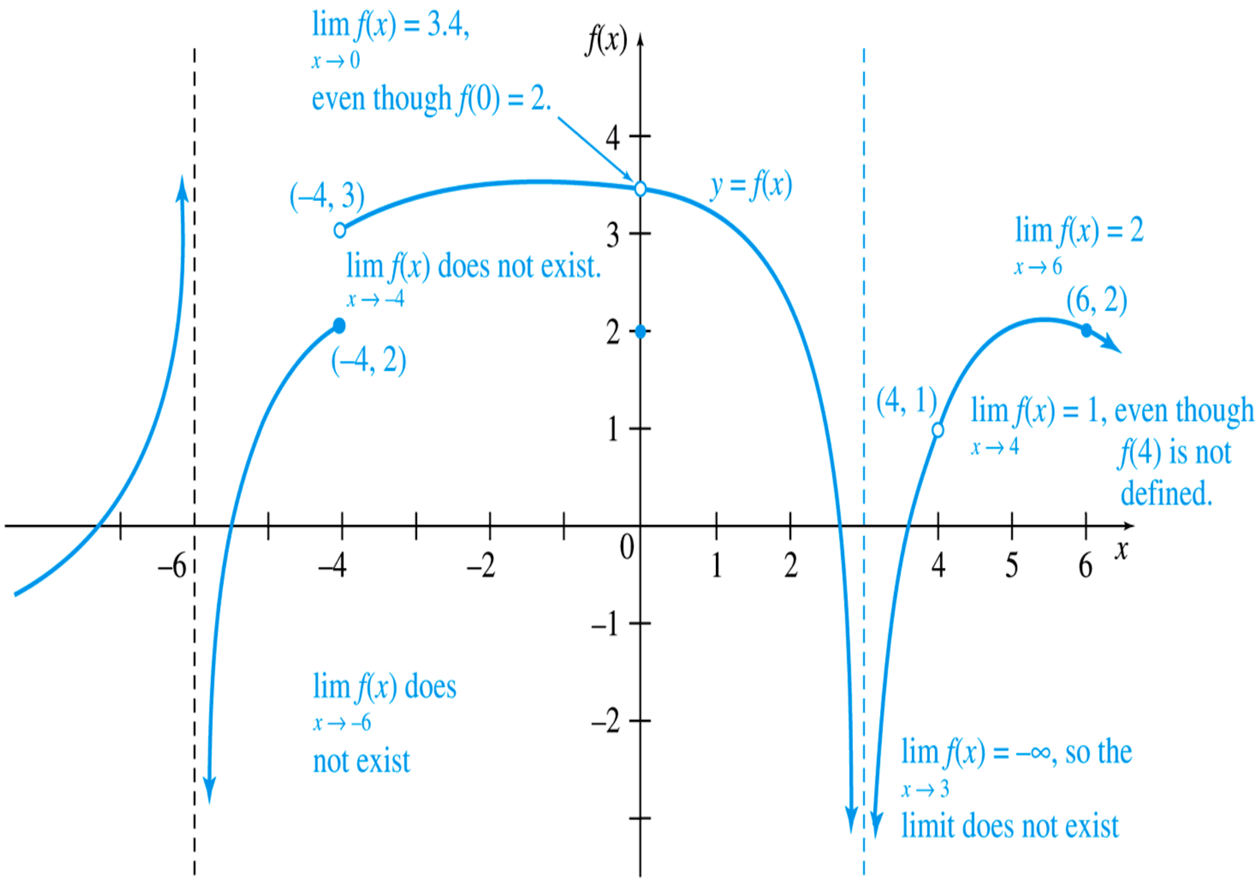
then *L* is the **limit** of $f(x)$ as *x* approaches *a*, written

$$\lim_{x \to a} f(x) = L.$$

EXISTENCE OF LIMITS

The limit of *f* as *x* approaches *a* may not exist.

1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as *x* approaches the number *a* from either side, we write $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$. In either case, the limit does not exist.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as *x* approaches *a* from one side and infinitely large in magnitude (negative) as *x* approaches *a* from the other side, then $\lim_{x \to a} f(x)$ does not exist.
3. If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \to a} f(x)$ does not exist.



WS – limits

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